AMS210.01.
Homework 7

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This homework is optional. You can solve it to get some extra-points in this class (contact me to get them). I recommend you to solve problems from it — this will really help you during your final!

1. Let the coordinates of the vector v in the "old" basis are $v = (6, 9, 14)$. Find the coordinates of the vector in the "new" basis $\{e_1 = (1, 1, 1), e_2 = (1, 1, 2), e_3 = (1, 2, 3)\}$.

2. Let the "old" basis is $\{e_1 = (1, 2, 1), e_2 = (2, 3, 3), e_3 = (3, 8, 9)\}$, and "new" basis is $\{e'_1 = (3, 5, 8), e'_2 = (5, 14, 13), e'_3 = (1, 9, 2)\}$. Find the change-of-basis matrix.

3. How does the change-of-basis matrix change if we

   (a) interchange two vectors of the "old" basis?

   (b) interchange two vectors of the "new" basis?

   (c) take vectors of both bases in reverse order?

4. Determine, which of the following mappings are linear operators. Justify your answer.

   (a) $x \mapsto a$, where a is a fixed vector.

   (b) $x \mapsto x + a$, where a is a fixed vector.

   (c) $x \mapsto \alpha x$, where α is a given number.

   (d) $x \mapsto \langle x, a\rangle b$, where a, b are fixed vectors, and V is a Euclidean space.

   (e) $f(x) \mapsto f(x + 2)$, where f is a function.

   (f) $f(x) \mapsto f(x + 1) - f(x)$, where f is a function.

   (g) $(x_1, x_2, x_3) \mapsto (x_1 + 2, x_2 + 5, x_3)$.

   (h) $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_1 + x_2 + x_3)$.

5. Find the matrix of the following linear operators:

   (a) $(x_1, x_2, x_3) \mapsto (x_1, x_1 + 2x_2, x_2 + 3x_3)$ in the standard basis.

   (b) Projection along the y-axis in the standard basis.

   (c) $x \mapsto (x, a)a$ in the standard basis of Euclidean space, if $a = e_1 - 2e_3$.

   (d) $X \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} X$ in the standard basis in the matrix space $M_{2,2}$.

(e) $X \mapsto X^\top$ in the standard bases in the space $M_{2,2}$.

(f) Differentiation in the space P_4 in the basis $\{1, t, t^2, t^3, t^4\}$.

(g) Differentiation in the space P_4 in the basis $\{t^4, t^3, t^2, t, 1\}$.

(h) Differentiation in the space P_4 in the basis $\{1, t, \frac{t^2}{2!}, \frac{t^3}{3!}, \frac{t^4}{4!}\}$.

(i) Rotation of the space by an angle $\frac{2\pi}{3}$ around the line given by $x_1 = x_2 = x_3$ in the standard basis.

6. Let the linear operator in the space V has the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 5 & 4 & 0 & -1 \\ 3 & 2 & 0 & 3 \\ 6 & 1 & -1 & 7 \end{pmatrix}$$

with respect to basis $\{e_1, e_2, e_3, e_4\}$. Find the matrix of this operator in the following bases:

(a) $\{e_2, e_1, e_3, e_4\}$

(b) $\{e_1, e_1 + e_2, e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_4\}$.

7. Let the linear operator in the space $\mathbb{R}^3$ has matrix

$$\begin{pmatrix} 1 & -18 & 15 \\ -1 & -22 & 20 \\ 1 & -25 & 22 \end{pmatrix}$$

with respect to basis $\{(8, -6, 7), (-16, 7, -13), (9, -3, 7)\}$. Find its matrix with respect to basis $\{(1, -2, 1), (3, -1, 2), (2, 1, 2)\}$.

8. Find characteristic polynomial, the eigenvalues and corresponding eigenvectors of the following matrices:

(a) $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix}$

9. Find the basis of the eigenspace associated with the given value of λ

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\lambda = 2$.

(b) $\begin{pmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\lambda = 2$.

10. Find the eigenvectors and eigenvalues of the following operators:

    (a) $x \mapsto \alpha x$, where α is a fixed number.

    (b) Differentiation in the space P_n.

    (c) $X \mapsto X^\top$ in the space of $n \times n$-matrices.

    (d) $f \mapsto x \frac{d^n f}{n x^n}$ in the space P_n.

11. Show that eigenvalues of A and $A^\top$ are the same.

12. (a) Prove that the eigenvector of the operator $\mathcal{A}$ corresponding to the eigenvalue λ is the eigenvector of $\mathcal{A}^2$ with the eigenvalue λ^2.

    (b) Prove that the eigenvector of the operator $\mathcal{A}$ corresponding to the eigenvalue λ is the eigenvector of $\mathcal{A}^n$ with the eigenvalue λ^n.

    (c) Prove that the eigenvector of the operator $\mathcal{A}$ corresponding to the eigenvalue λ is the eigenvector of $f(\mathcal{A})$ with the eigenvalue $f(\lambda)$, where f is an arbitrary polynomial.

13. The matrix A is called **nilpotent** if $A^k = 0$ for some k. Show that if A is nilpotent, then 0 is the only eigenvalue of A.

14. Prove that if $\mathcal{A}^2$ has an eigenvalue λ^2, then either λ or $-\lambda$ is an eigenvalue for $\mathcal{A}$.

15. (a) Show that $\det A$ is equal to the product of all eigenvalues of A.

    (b) Show that A is non-invertible, if and only if 0 is an eigenvalue of A.

16. Let λ be an eigenvalue of the invertible matrix A with associated eigenvector x. Show that $1/\lambda$ is an eigenvalue for A^{-1} with the associated eigenvector x.

17. (a) Show that the coefficient before λ^{n-1} in the characteristic polynomial of A is equal to the $-\operatorname{tr} A$.

    (b) Show that $\operatorname{tr} A$ is equal to the sum of of the eigenvalues of A.

18. Let λ be an eigenvalue of A with the associated eigenvector x. Show that $\lambda + c$ is an eigenvalue for $A + cI$ with the associated eigenvector x.

19. Determine whether the following matrices are diagonalizable. If they are diagonalizable, find their diagonal form and corresponding basis.

    (a) $\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$

    (b) $\begin{pmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

20. Find a 2×2 non-diagonal matrix whose eigenvalues are 2 and -3 and associated eigenvectors are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

21. (a) Compute $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}^{50}$

    (b) Compute $\begin{pmatrix} 7 & -4 \\ 14 & -8 \end{pmatrix}^{64}$

22. Solve the equations:

    (a) $X^2 = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$

    (b) $X^2 = \begin{pmatrix} 6 & 2 \\ 3 & 7 \end{pmatrix}$

23. Show that if A is diagonalizable, then

    (a) $A^\top$ is diagonalizable.

    (b) A^k is diagonalizable.

    (c) if A is invertible, then A^{-1} is diagonalizable.

24. Let A be a $n \times n$-matrix, and let $B = P^{-1}AP$. Show that if x is an eigenvector of A associated with the eigenvalue λ, then $P^{-1}x$ is an eigenvector of B associated with the eigenvalue λ.

25. Prove that characteristic polynomials of AB and BA are equal (A and B are 2 square equal-sized matrices).

26. Let

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} (a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1^2 & a_1a_2 & \dots & a_1a_n \\ a_2a_1 & a_2^2 & \dots & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_na_1 & a_na_2 & \dots & a_n^2 \end{pmatrix}$$

    Find its eigenvalues.

27. Prove that any polynomial of degree n is a characteristic polynomial of some matrix.